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Field Theories and Growth Models

Giorgio Parisi[1,2] and Zhang Yi-Cheng[3]

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Continuum field theories for the Eden and DLA models are formulated, and they are shown to be related to the reggeon field theories with local and non-local interactions, respectively.

KEY WORDS: Eden model; DLA; growth processes; field theories; mean field approximations.

1. INTRODUCTION

Kinetic growth processes have attracted much interest recently. Notably the diffusion limited aggregation model[1] (DLA) and the Eden model.[2] In a broader sense these processes can be found in studies of biological pattern formations,[3] chemical reactions,[4] epidemic process with immunization,[5] solidification instabilities,[6] directed percolation,[7] and automata,[8] just to mention a few of the general examples of these growth processes which are Markovian (or quasi-Markovian) and irreversible. In this paper we will concentrate mainly on the Eden model and a section on the DLA. The Eden model describes a growth process in which when a seed is introduced it grows by creating new particles in its neighborhood. DLA is different from the Eden model in that a new particle can be created only by capturing a diffusing particle which is performing random walk in the neighborhood of the growing objects. Most recent works are based on numerical simulations and mean field type equations, with the exception that the exact solution for the Eden model with large dimensionality has been found.[9]

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[1] University Tor Vergata, Roma, Italy.
[2] INFN, Roma, Italy.
[3] Physics Department, Brookhaven National Laboratory, Upton, New York 11973.

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Following a well-established tradition (see Ref. 5 and references therein) in this paper we would like to give a field theory formulation for the above two problems, which can only be an approximation to the discrete model. However, this approximation is expected to be good when the scale of interest is large. The advantage of having such a field formulation is clear: it would serve as the starting point for the mean field approximation and take into account systematically the fluctuations. One might also be able to develop some appropriate renormalization techniques.

Reggeon field theory (RFT) was originally developed for analyzing high-energy scattering amplitudes (for a review cf. Ref. 10). It was realized some years ago that RFT actually describes Markovian process,[11] and it was also discovered that directed percolation and an epidemic process with immunization are related to RFT.[5,7]

In Section 2 we give a heuristic derivation of a RFT-type theory for the Eden model; in Section 3 we study the mean field equations and their stabilities, in Section 4 we give a similar analysis for DLA, and many open questions are briefly discussed in Section 5. In principle we could also argue that the Eden model and directed percolation are related (directed percolation may be interpreted as an epidemic problem[7]) and use the established relations between RFT and directed percolation, however we think that it is more instructive to present a self-contained derivation.

2. FIELD THEORETICAL FORMULATION FOR THE EDEN MODEL

The Eden growing process is the simplest, so we will start with the Eden model. The Eden model can be briefly described as follows. We introduce a seed particle at the origin of the d-dimensional Euclidean space at time zero, and allow this seed particle to grow by creating new particles at its empty neighbors in the next time step, with the total probability normalized to one. This process will continue as the time t increases. At $t = n$ we have a whole ensemble of clusters of n particles with various weights. It is possible to perform an exact counting of these clusters in infinite dimensions as well as to compute the first $1/d$ correction.[9] However in finite dimensions things are more complicated and we are thus motivated to develop a kind of field theory to describe the Eden model, hoping to shed more light on this model, in a restrictive sense, and on the kinetic growth processes in a broader sense. Later we shall see that the field formulation which is developed below happens to be the well-studied reggeon field theory in high-energy physics.

We define a series order parameters for the Eden model:

$$
\begin{aligned}
&\rho_1(x; n)\\
&\rho_2(x_1, x_2; n)\\
&\rho_3(x_1, x_2, x_3; n)\\
&\vdots\\
&\rho_l(x_1, x_2, \ldots, x_l; n)
\end{aligned}
\tag{1}
$$

They are interpreted as the probability of finding a particle at x at time step n, the joint probability of finding particles at x_1 and x_2 at time step n, and so on.

Generally speaking, to each cluster C of n particles we associate a density function $\rho(x)$ which is equal to the number of particles at the point x. [In the Eden model $\rho(x)=0$ or 1, but it is convenient to consider also more complicated models.] The precise definition of the ρ_l is

$$
\begin{aligned}
\rho_1(x; n) &= \langle \rho(x) \rangle_n\\
\rho_2(x_1, x_2; n) &= \langle \rho(x_1)\, \rho(x_2) \rangle_n
\end{aligned}
$$

where the average is done on all the clusters of n-particles generated according to the appropriate dynamical law. Our aim is to write down the equations for the ρ's for the stochastic processes which are similar to the Eden model or to the DLA model. All the main ingredients of the original model should be preserved in order not to change the universality class.

The main characteristics of the Eden model are diffusion and self-inhibition. Let us study a simple model with only diffusion, i.e., a site may become occupied only if a nearby site is occupied. Since no inhibition is present, multiple occupancy is permitted. In the simplest version of the model the probability of adding a new particle is proportional to the occupancy of the nearby sites. In this case the equation for ρ_1 is rather simple:

$$
\rho_1(x, n+1) - \rho_1(x, n) = \frac{1}{2dn} \sum_{u=1}^{d} [\rho_1(x+\hat{\mu}, n) + \rho_1(x-\hat{\mu}, n)] \tag{2}
$$

This equation expresses the fact that the increase in probability of finding a particle at x is due to occupancy of its neighbors at the previous time step. Since it is assumed to be an n-particle object, and since every particle can grow, we normalize the increase by a factor $1/2dn$. Later on we shall see that n itself should be renormalized.

It may be convenient to write Eq. (2) in the continuum notation. For this purpose we define $t = \ln(n)$ and then Eq. (2) becomes

$$\dot{\rho}_1(x; t) = \alpha \Delta \rho_1(x; t) + \rho_1(x; t) \tag{3}$$

where $\alpha = 1/2d$, Δ being the lattice Laplacian. We see that if initially $\rho_1(x, 0) = 0$ then $\rho_1(x, t)$ will remain zero forever. We have to add a seed particle at the beginning, and Eq. (3) then becomes

$$\dot{\rho}_1(x; t) = \alpha \Delta \rho_1(x; t) + \rho_1(x; t) + J\delta(t)\,\delta(x) \tag{4}$$

where J is the strength of the source (later on we shall put it to be one or otherwise specified). If we approximate the lattice Laplacian with the continuum one, the solution of Eq. (4) is well known:

$$\rho_1(x; t) \sim \frac{1}{(\alpha t)^{d/2}} \exp\left(t - \frac{x^2}{4\alpha t}\right) \tag{5}$$

The diverging factor reflects the ever-growing fact that

$$n = \int dx \rho_1(x; t) = \exp(t) \tag{6}$$

(as it should be) and that

$$\langle x^2 \rangle = \int dx \rho_1(x; t)\, x^2 \sim t = \ln(n) \tag{7}$$

We recover in this way the results for the infinite dimensional Eden model where self-inhibition is not relevant.

We must be slightly more careful in performing the continuum limit: Eq. (4) has been written in lattice spacing units and the coefficients of $\Delta\rho_1$ and ρ_1 are fixed and cannot be varied. Indeed, after the appropriate redefinitions of scales Eq. (4) becomes

$$\dot{\rho}_1(x; t) = \Delta \rho_1(\lambda; t) + \frac{2d}{a^2} \rho_1(x; t) + J'\delta(t)\,\delta(x) \tag{8}$$

which clearly has problems with the limit $a \to 0$. If we generalize the model by assuming that at each step in time a particle may be taken out with a probability g, and a particle may be added with the same probability as in the previous example, Eq. (8) becomes

$$\rho_1(x; t) = \Delta \rho_1(x; t) + \frac{2d - g}{a^2} \rho_1(x; t) + J'\delta(t)\,\delta(x) \tag{9}$$

The continuum limit in (9) can be reached now by sending a to zero and g to $2d$ simultaneously. Therefore the continuum limit may be reached only by introducing a carefully balanced process of "birth" and "death" of particles. We note that although the relation among t and n is not so simple as before the relation $\langle x^2 \rangle \simeq \ln(n)$ still holds. In other words, we stay in the same unversality class as far as the shape of clusters at large time is concerned. We hope that similar modifications in more complex cases will not change the universality class.

If we want to develop something similar to the Eden model we need to introduce a mechanism of inhibition. The simplest such mechanism is to let the probability of adding a particle at the point x be given by $\sum_\mu [\rho(x+\hat{\mu}) - g\rho(x)]$. (A negative probability for adding is interpreted as a probability for removing.) With this hypothesis we propose the following equation:

$$\dot{\rho}_1(x;t) = \alpha \Delta \rho_1(x;t) + \rho_1(x;t) - g^2 \rho_2(x,x,t) + J\delta(t)\,\delta(x)$$

The above equation is actually a truncated one; it might have well included $\rho_3(x,x,x;t)$ and so on. Here we stick to the simplest assumption that the inhibition is due only to two-body interactions. To have a complete set of equations for $\rho_2, \rho_3, \cdots$ we suggest that the following set of equations approximately describes the Eden model:

$$\dot{\rho}_1(x;t) = \alpha \Delta \rho_1(x;t) + \rho_1(x;t) - g^2 \rho_2(x,x;t) + \delta(t)\,\delta(x) \quad (10a)$$

$$\begin{aligned} \dot{\rho}_2(x_1,x_2;t) = {} & \alpha(\Delta_1 + \Delta_2)\,\rho_2(x_1,x_2;t) + 2\rho_2(x_1,x_2;t) \\ & - g^2[\rho_3(x_1,x_1,x_2;t) + \rho_3(x_1,x_2,x_2;t)] \\ & + \delta(x_1 - x_2)[\rho_1(x_1;t) + \rho_1(x_2;t)] \end{aligned} \quad (10b)$$

where Δ_1 and Δ_2 are the Laplacians at $\mathbf{x}_1$ and x_2, respectively, and so on, this is an infinite set of equations. Note that $\rho_2(x_1,x_2;t)$, $\rho_3(x_1,x_2,x_3;t)$ and so on are symmetric functions, and note that in Eq. (10b) the source term is due only to the presence of $\rho_1(x;t)$. By definition, the functions ρ_2, ρ_3 and so on can be decomposed into connected parts and factorized parts. For example,

$$\rho_2(x_1,x_2;t) = \rho_1(x_1;t)\,\rho_1(x_2;t) + \rho_2^c(x_1,x_2;t) \quad (11a)$$

$$\begin{aligned} \rho_3(x_1,x_2,x_3;t) = {} & \rho_1(x_1;t)\,\rho_1(x_2;t)\,\rho_1(x_3;t) \\ & + \rho_1(x_1;t)\,\rho_2^c(x_2,x_3) \\ & + \rho_1(x_2)\,\rho_2^c(x_1,x_3) + \rho_1(x_3)\,\rho_2^c(x_1,x_2) \\ & + \rho_3^c(x_1,x_2,x_3) \end{aligned} \quad (11b)$$

The theory described by Eq. (10) is not deterministic but stochastic.

Now we turn our attention to a reggeon field theory. The action is

$$S=\int dxdt[\phi^{+}\partial_{t}\phi+\alpha\nabla\phi^{+}\nabla\phi-\phi^{+}\phi-ig\phi^{+}(\phi^{+}+\phi)\,\phi] \tag{12}$$

and

$$Z=\int D\phi D\phi^{+}e^{-S}$$

We make an identification between the two theories as follows:

$$\rho_n(x_1, x_2,..., x_n; t)=\frac{1}{(ig)^{n-1}}\langle\phi(x_1, t)\,\phi(x_2, t)\cdots\phi(x_n, t)\,\phi^{+}(0, 0)\rangle \tag{13}$$

Here $\langle\quad\rangle$ denotes the average using Eq. (12). The right-handside of Eq. (13) means that when a seed particle is introduced at (0, 0) we find particles simultaneously at $x_1, x_2,..., x_n$ at time t. It is illustrated by writing down the Feynman diagrams. In configuration space we have the following diagrams for ρ_1, ρ_2, and ρ_3(see Fig. 1). The intermediate coordinates $x', t',...$ must be integrated over. In Fig. 1 only tree diagrams are drawn, and diagrams dressed with all possible loops should be added to them, just as in usual perturbation theory.

We are left to establish the correspondence in Eq. (13). We could demonstrate it via a perturbation expansion, as we were originally led to it. However, here we will show it using the formal Dyson–Schwinger equation approach, which is heuristic but more compact. Let us consider the following equation:

$$0=\int D\phi D\phi^{+}\frac{\delta}{\delta\phi^{+}(x, t)}\phi^{+}(0, 0)\,e^{-S} \tag{14}$$

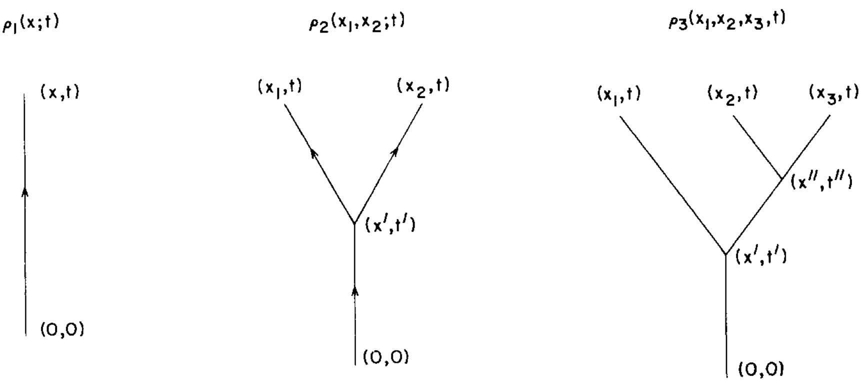


Fig. 1. Tree diagrams for the density functions ρ_1, ρ_2, and ρ_3.

where S is the action given by Eq. (12). This is zero because it is a total functional derivative. Performing the functional derivative we arrive at the equation

$$\partial_t \langle \phi(x, t)\, \phi^+(0, 0)\rangle = \alpha \Delta \langle \phi(x, t)\, \phi^+(0, 0)\rangle + \langle \phi(x, t)\, \phi^+(0, 0)\rangle + \delta(x)\, \delta(t) + ig\langle \phi^2(x, t)\, \phi^+(0, 0)\rangle \qquad (15)$$

With the help of Eq. (13) the reader will recognize this as Eq. (10a). Note that we have set $\langle \phi(x, t)\, \phi^+(x, t)\, \phi^+(0, 0)\rangle$ to zero, as can be explicitly checked.

For Eq. (10b) we can consider the following equation:

$$0 = \int D\phi D\phi^+ \left[\frac{\delta}{\delta\phi^+(x_1, t)}\, \phi^+(0, 0)\, \phi(x_2, t) + \frac{\delta}{\delta\phi^+(x_2, t)} \times \phi^+(0, 0)\, \phi(x_1, t)\right] e^{-S} \qquad (16)$$

carrying out the functional derivatives we get the equation

$$\begin{aligned}\partial_t \langle \phi(x_1, t)\, \phi(x_2, t)\, \phi^+(0, 0)\rangle = {} & \alpha(\Delta_1 + \Delta_2)\langle \phi(x_1, t)\, \phi(x_2, t)\, \phi^+(0, 0)\rangle \\ & + 2\langle \phi(x_1, t)\, \phi(x_2, t)\, \phi^+(0, 0)\rangle \\ & + ig(\langle \phi(x_2, t)\, \phi^2(x_1, t)\, \phi^+(0, 0)\rangle \\ & + \langle \phi(x_1, t)\, \phi^2(x_2, t)\, \phi^+(0, 0)\rangle) \qquad (17)\end{aligned}$$

where we have used the fact that $\langle \phi(x, t)\rangle = \langle \phi^+(x, t)\rangle = 0$ (we shall discuss the case of the nontrivial vacuum in Section 3) and we have set $\langle \phi^+(x, t) \cdots \phi^+(0, 0)\rangle$ to zero corresponding the fact only one seed was introduced at (0, 0). A physically motivated boundary condition like the last piece of Eq. (10b) has to be added before one can realize that Eqs. (17) and (10b) are actually equivalent with the help of Eq. (13).

The Dyson–Schwinger equation approach can be carried on and on. It provides the formal evidence that the theory described by the infinite set of equations Eq. (10) can be formulated by RFT equations [(12), (13)]. As we have discussed we can modify the coefficient of $\phi^+\phi$ by adding a probability for removing a particle.

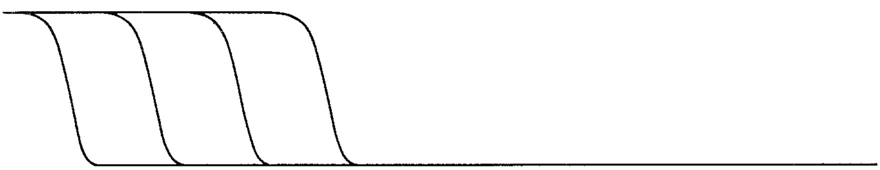
3. MEAN FIELD SOLUTION AND ITS INSTABILITIES

In this section we discuss the mean field approximation to the theory in Eq. (10). It may also be thought of as a classical approximation when fluctuations are not important. We considèr Eq. (10a), in the present case we can take $\rho_2 \sim \rho_1^2$

$$\dot{\rho}_1(x; t) = \alpha\Delta\rho_1(x; t) + \mu^2\rho_1(x; t) - g^2\rho_1^2(x; t) + \delta(x)\,\delta(x) \qquad (18)$$

where an arbitrary mass μ is introduced to remind us of the correct dimensionality. Initially we set $\rho_1(x, t) = 0$ everywhere which corresponds to the vacuum. However we see this vacuum is unstable against any perturbation, such as putting a seed at (0, 0). The region perturbed by the external source soon reaches a stable value $\rho_1 = \mu^2/g^2$, and perturbations (external or internal) around this value are stable in the sense that they disappear with increasing time. We present the numerical solutions for the one-dimensional and two-dimensional cases in Fig. 2, where one can see the

a



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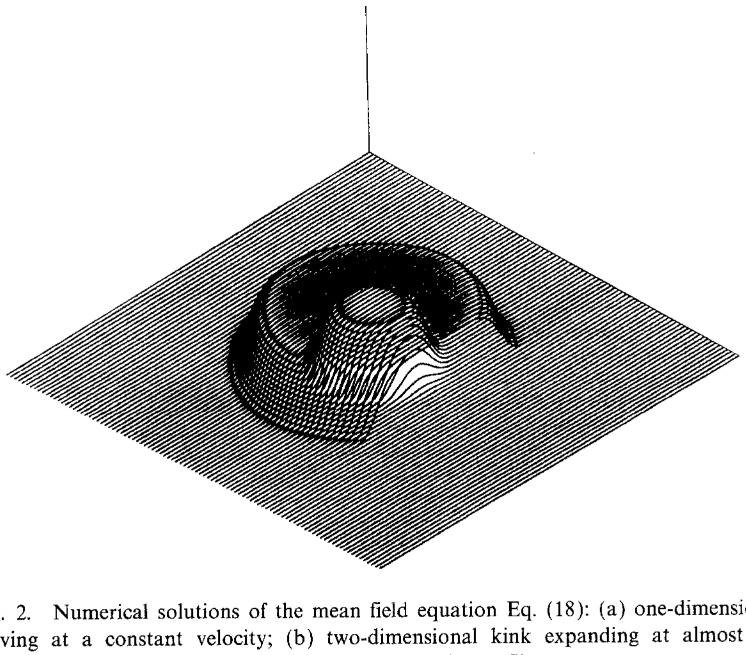

Fig. 2. Numerical solutions of the mean field equation Eq. (18): (a) one-dimensional kink moving at a constant velocity; (b) two-dimensional kink expanding at almost constant velocity, we stop plotting the larger kink to show the profile.

time evolution of the solution to Eq. (18). The solutions appear to move with a constant velocity and an invariant moving kink front. The regions which have reached the stable value μ^2/g^2 can be interpreted as where growth has taken place, and they have unity particle density.

Regretably it is not possible to solve analytically the nonlinear equation Eq. (18) even for the one-dimensional case. However, we may determine the moving kink velocity by various methods. One which is particularly illustrative was introduced by Dee and Langer[12] for a similar case. Let us concentrate on the one-dimensional case. From Fig. 2 we can infer that $\rho_1(x, t)$ can be written

$$\rho_1(x, t) = f(x - vt) \tag{19}$$

Equation (18) becomes for some finite time

$$\alpha f'' + vf' + \mu^2 f - g^2 f^2 = 0 \tag{20}$$

There is a mechanical analog to this quation, in which f may be interpreted as the position of the particle and x as time. Equation (20) then describes the motion of an anharmonic damped oscillator in a potential

$$-V = \frac{1}{2}\mu^2 f^2 - \frac{g^2}{3} f^3 \tag{21}$$

which is shown in Fig. 3. The particle initially stays on top of the hill. Negative f would mean negative particle density in the original problem

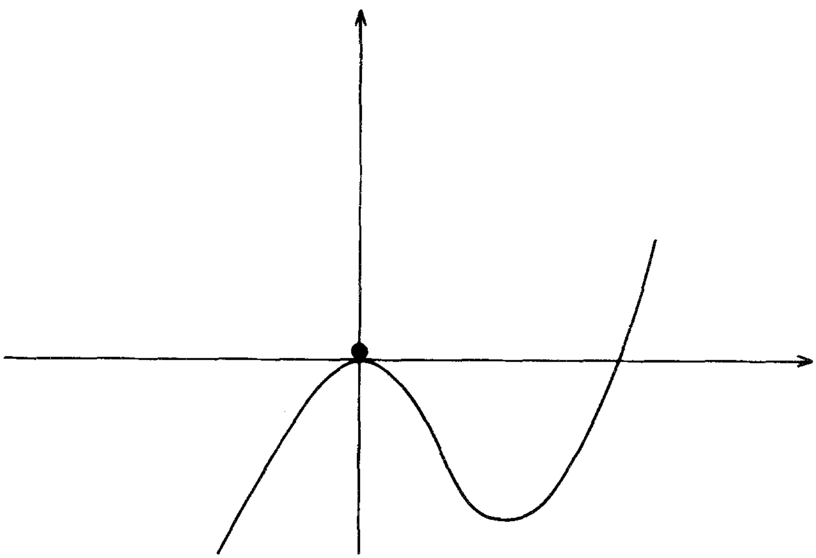

Fig. 3. The potential of a mechanical analogy.

and is thus unphysical. Under a small positive perturbation the particle would run down to the bottom of the potential valley. There are various ways to reach the bottom and it would be sufficient to study the motion in the vicinity of the bottom. For this reason we can linearize Eq. (20) near the stable value with the replacement

$$f=\frac{\mu^2}{g^2}+\delta \tag{22}$$

We obtain the equation

$$\alpha\delta''+v\delta'-\mu^2\delta=0 \tag{23}$$

Note that the sign of $\mu^2\delta$ is opposite to what it "should" be since we are dealing only with a mechanical analog. We assume that among all possible ways of descending (i.e., oscillating and overdamping solutions) the particle tries to follow the way which needs least time. It is not difficult to find, using the linearized Eq. (23), that there is a critical $v_c=2(\alpha\mu)^{1/2}$ which separates the overdamped solution from the oscillating one. This velocity is known to be the velocity of propagation of the front. (See Dee and Langer[12] and Langer[6] and references therein.) In general the velocity of propagation of the kink front may be selected using the criterion of marginal stability. To apply it in this case we write

$$\begin{gathered}\rho(x,t)=f(x-vt)+\omega(x-vt)\\ \omega\ll 1\end{gathered} \tag{24}$$

ω satisfies the differential equation

$$\alpha\omega''+v\omega'+\mu^2\omega-2g^2\omega f\equiv\mathscr{D}\omega=0 \tag{25}$$

It is interesting to consider the eigenvalues λ of $\mathscr{D}$

$$\mathscr{D}\omega_n=\lambda_n\omega_n \tag{26}$$

If no λ_m is positive and only one is zero, the solution is said to be marginally stable. (If all are strictly negative the solution is stable, while if some λ_m is strictly positive then the solution is unstable.) The marginal stability criterion states that in general the kink front velocity is selected by imposing that the solution is marginally stable. Here we notice that if the equation we consider is the mean field approximation of a stochastic process the marginal stability criterion comes out quite naturally.

To be explicit let us consider our one-dimensional "Eden model": As the procedure of adding particles will be a stochastic one we expect that the velocity of propagation of the front will fluctuate once in a while. If, as it is natural, these fluctuations are uncorrelated in time the position of the kink after a large time t will be

$$x(t) = vt + v\sqrt{t}\cdot r \tag{27}$$

where r is a Gaussian random number with variance 1 and 0 mean. Let us compute the expectation values of ρ_1 and ρ_2. If

$$\rho_1 = \rho_2 \simeq 1 \qquad \text{for} \quad x \ll vt$$

$$\rho_1 = \rho_2 \simeq 0 \qquad \text{for} \quad x \gg vt$$

we have that

$$\rho_1 = \rho_2 = \tfrac{1}{2} \qquad \text{for} \quad x = vt \tag{28}$$

In other words the corrected correlation function $\rho_1^2 - \rho_2$ is not small near vt! It is easy to see that ρ_2 is substantially different from 0 in the region $x - vt = O(v\sqrt{t})$. These results imply that the corrections to the mean field equations cannot be small. However they are not arbitrarily large as they increase as $\sqrt{t}$.

If we come back to the scheme described in the previous section we have that the first correction to the mean field approximation is given by assuming that only ρ_2^c is different from zero. Under the hypothesis that ρ_2^c is small we have

$$\begin{aligned}\dot{\rho}^c(x_1, x_2) = (\Delta_1 + \Delta_2)\,\rho^c + 2\mu^2\rho^c(x_1, x_2)\\ - 2g[\rho_1(x_1) + \rho_2(x_2)]\,\rho_2^c(x_1, x_2)\end{aligned} \tag{29}$$

plus source terms.

If we look for solutions of (29) in the form

$$\rho_2^c(x_1, x_2; t) = g(vt - x_1)\, g(vt - x_2) \tag{30}$$

we see that g satisfies the equation

$$\mathscr{D}g = \text{source terms}$$

which cannot be solved when $\mathscr{D}$ has a zero eigenvalue. In other words the marginal stability condition implies that the connected correlation function are not small. This scenario is well known in field theory (where the zero mode always exists due to Galilei invariance), and the existence of a zero mode is necessary to restore the quantum diffusion of the solution. The standard way to cope with these problems is the following. While standard mean field theory consists of using the saddle point method for evaluating the functional integral, in the presence of a zero mode the saddle point variety is larger and collective coordinates must be introduced.[4] In this way it should be possible to find the diffusion constant v of Eq. (27). To study this in detail goes beyond the aims of this paper.

4. DLA MODEL

Much attention has been given recently to the diffusion limited aggregation (DLA) model proposed by Witten and Sander.[1] It has been studied extensively using Monte Carlo simulations[1,13] and mean field equations.[14] The original DLA model is stochastic, and one would like to write the DLA theory in a continuum version, at least approximately. In this section we attempt only a heuristic derivation. We shall see the resulting field theory is a modified RFT with nonlocal interactions.

Let us consider the simplified DLA mean field equation

$$\dot{\rho}(x) = p(x)[\Delta\rho(x) + \rho(x)] \tag{31a}$$

$$\Delta p(x) = g^2 p(x)(\Delta\rho + \rho) \tag{31b}$$

where $\rho(x)$ is the particle density of the growing cluster, $p(x)$ is the probability of a random walking particle encountering the cluster, and all nonessential parameters have been set to 1. Note that growth can only take place by capturing the random walking particles. Define

$$\tilde{p}(x) = 1 - p(x) \tag{32}$$

which represents the penetration probability and goes to zero as x goes to infinity. The solution of Eq. (31b) is

$$\tilde{p}(x) = g^2 \int K(x, y)[\Delta\rho(y) + \rho(y)]\, dy \tag{33}$$

where

$$[\Delta_x - g^2(\Delta\rho + \rho)]\, K(x, y) = \delta(x - y)$$

This can be written formally as

$$\tilde{p}(x) = \frac{-g^2}{\Delta - g^2(\Delta\rho + \rho)} (\Delta\rho + \rho)$$

[4] For a review see Ref. 16.

or

$$p(x)=\frac{1}{1-\frac{1}{\Delta}g^2(\Delta\rho+\rho)} \tag{34}$$

The final evolution equation for ρ is given by

$$\begin{aligned}\dot{\rho}(x)&=[\Delta\rho(y)+\rho(x)]\left\{1+g^2\frac{1}{\Delta}[\rho(x)+\Delta\rho]\right.\\&\left.\quad+\ g^4\frac{1}{\Delta}(\Delta\rho+\rho)\frac{1}{\Delta}(\Delta\rho+\rho)+\cdots\right\}\\&=[\Delta\rho(x)+\rho(x)]\frac{1}{1-\frac{g^2}{\Delta}[\rho(x)+\Delta\rho]}\end{aligned} \tag{35}$$

The above equation can be considered as a "fractorized" mean-field approximation. To restore the stochastic model, we follow the same steps as in the Eden model, i.e., we consider the model in which we add a particle at the point x with probability $\rho(x)[\Delta\rho(x)+\rho(x)]$ and we obtain the following equations:

$$\begin{aligned}\dot{\rho}_1(x)&=(\Delta_x+1)\left[\rho_1(x)-g^2\int dyG(x-y)(1+\Delta_y)\,\rho_2(x,y)+\cdots\right]\\&\quad+\ \delta(t)\,\delta(x)\\\dot{\rho}_2(x_1,x_2)&=(\Delta_1+\Delta_2+2)\left\{\rho_2(x_1,x_2)-g^2\int dy[G(x_1-y)+G(x_2-y)]\right.\\&\quad\left.\times\ (1+\Delta_y)\,\rho_3(x_1,x_2,y)+\cdots\right\}+\delta(x_1-x_2)[\rho_1(x_1)+\rho_1(x_2)]\end{aligned} \tag{36}$$

where

$$\Delta_x G(x)=-\delta(x) \tag{37}$$

Equations (36), (37) correspond to a real DLA model in which we take into account fluctuation effects. The corresponding RFT action is

$$S=\int d^dx\,dt\{\varphi^+[\partial_t\varphi-i[1+\Delta^{-1}g^2(\varphi+\alpha\Delta\varphi)]^{-1}(\alpha\Delta\varphi+\mu^2\varphi)-i\varphi^+\varphi]\} \tag{38}$$

with the identification Eq. (13). One can expand the nonlocal interaction term and if higher powers of ϕ are not important at large distances then we have

$$S=\int d^dx\,dt\Bigg[\phi^+\partial_t\phi+\alpha\nabla\phi^+\nabla\phi+\mu^2\phi^+\phi+ig(\alpha\nabla\phi^+\cdot\nabla\phi-\phi^+\phi)$$
$$\times\int d^dyG(x-y)(\alpha\Delta+\mu^2)\,\phi(y)-ig\phi^+\phi^+\phi\Bigg] \tag{39}$$

We have arrived at a RFT with the nonlocal interaction.

The original Eden model corresponds to the theory with $-\mu^2$, and taking a smaller value of μ^2 corresponds to adding a probability for a particle at a given site to disapear. In other words a particle coming from infinity may hit the cluster and not be adsorbed, removing a particle from the cluster. Hopefully as soon as μ^2 is negative (supercritical region) we should be in the same universality class as standard DLA as far as the large time behavior is concerned.

In principle there is a critical value of μ^2 (μ_c^2) such that for $\mu^2<\mu_c^2$ no front is created and the effect of the original seed disappears. Using renormalization group arguments we could derive a scaling law for μ^2 near μ_c^2, e.g.,

$$\langle x^2\rangle=\frac{1}{|\mu^2-\mu_c^2|^\gamma}f\left(\frac{t^\beta}{\mu^2-\mu_c^2}\right)$$

γ and β being appropriate critical exponents. Such scaling laws are well known for the Ising model where γ and β can be computed in powers of $\varepsilon=4-d$, d being the space dimension. It would certainly be interesting to see if a similar computation can be done for the DLA model.

5. DISCUSSION

The field theory formalism we have studied for the Eden model and for the DLA gives us the possibility of controlling corrections to the mean field theory due to fluctuations. If we introduce in both cases a parameter which controls the growth, these corrections will be very important near the critical value of this parameter. An open problem is the behavior of the Eden and DLA models when the cluster is growing forever. While in the Eden model it was possible to find a marginally stable spherically symmetric solution this is not possible for the DLA. The spherically symmetric solution is unstable. Moreover the predictions of the spherically symmetric mean field theory are definitely wrong, at least at low dimensions.

It is not clear what role fluctuations play in the large time behavior of the DLA. In the first scenario, if one studies a nonspherically symmetric solution of the mean field equation, the large time behavior of $\int dx\rho(x)$ and $\int dx\rho(x)\,x^2$ does not depend on the initial seed (which is not spherically symmetric), and indeed it is the correct time dependence. This case corresponds to the existence of "turbulent" solutions of the DLA equations, and fluctuations of stochasticity do not play an important role.

In the second scenario we may have that $\int dx\rho(x)$ and $\int dx\rho(x)\,x^2$ are independent of the initial seeds with probability one (we do not consider the case in which these two quantities depend on the initial seed) at large time but they give the wrong result. Indeed if we average over all the possible solutions of the mean field equation by changing the seed we introduce a well-defined measure on the space of solutions; on the contrary if we take the case of fluctuations, fluctuations will weigh differently for different classical trajectories and it is possible that the measure on the space of solutions will be quite different from the previous one.

It is clear that careful numerical experiments may discriminate between the two scenarios. From the theoretical point of view the next step should be to apply to DLA all the weapons that have been developed in the study of turbulence.(17) In particular it should be very interesting to develop the equivalent of the Kalmogorof laws.

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Note Added in Proof. After completion of this paper, we learned that L. Peliti has developed a similar formalism for the Eden and DLA models. He uses Fock space techniques which are more rigorous.(18) Using a Hamiltonian approach we can describe the Eden process by a field theory which is local in space but nonlocal in time.(19)

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